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A new kind of two-parameter deformation of Heisenberg and parabose algebras and related deformed derivative

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Abstract
We propose a new kind of two-parameter (p, q)-deformed Heisenberg and parabose algebra, which reduces to the Heisenberg algebra for the $p = 1$ case and to parabose algebra for $q = -1$ case. Corresponding to the two-parameter deformed oscillator, we also introduce a new kind of (p, q)-deformed derivative which relates to the ordinary derivative and q-deformed derivative in an explicit manner. We study the structure of Fock-like space of the new (p, q)-deformed oscillators and derive a formal solution for the eigenvalue equation of the Hamiltonian.

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1. Introduction

Over the past decade and more there has been a lasting interest in the study of quantum groups and algebras [1–4]. This is connected with the fact that these new mathematical structures are relevant for various problems in theoretical physics, such as the quantum inverse scattering method, exactly solvable statistical models, fractional statistics, and so on. It is also well known that parastatistics were introduced by Green [5] who observed that it is not necessary for all particles in nature to be either bosons or fermions because the principle of quantum field theory allows a more general statistics which includes the usual ones as its limiting cases.

On the other hand, deformed Heisenberg algebra with reflection (or R-deformed Heisenberg algebra (RDHA)) appeared in the context of Wigner's generalized quantization schemes [6] underlying the concept of parafields and parastatistics of Green, Volkov, Greenberg and Messian [7, 8]. The RDHA represents, probably, one of the first examples of the deformation of an ordinary harmonic oscillator which, as it was shown recently [9], possesses some universality being also related to parafermions, to (2 + 1)-dimensional anyons and to the bosonized form of supersymmetric quantum mechanics [10–12]. Besides, the RDHA structure underlies the construction of fractional supersymmetry [13, 14].

In this work, we generalize the RDHA to a deformed Heisenberg algebra with dilation by introducing a (p, q)-deformed derivative $D_{p,q}$. This generalization also has an intimate relation to parabose algebra, because when the dilation parameter is $q = -1$, the deformed Heisenberg algebra with dilation will reduce to the parabose algebra. So it can also be regarded as a kind of q-deformed parabose algebra, however, different from other q-deformed parabose algebras reported in the literature [15, 20, 30, 31]. The new deformations can be viewed as a mixture of the q-deformed Heisenberg algebra [16] and the classical non-deformed Heisenberg algebra, because the (p, q)-deformed derivative $D_{p,q}$ is a combination of the Jackson q-difference operator and the usual derivative satisfying with the coordinate operator, respectively, the q-deformed and the non-deformed Heisenberg commutation relations. It could also be of interest to mention here that the operator $D_{p,q}$ in the paraboson case when $q = -1$ after minor change of the deformation parameters is exactly a one-dimensional Dunkl operator. In this sense, $D_{p,q}$ can be regarded as a q-deformation of the Dunkl operator. The one-dimensional Dunkl operator is the first one in the family of multi-dimensional Dunkl differential–difference operators associated with reflection groups, the operators attracting great interest due to their important role in the harmonic analysis on Riemannian symmetric spaces and in the analysis of the quantum many body systems of Calogero–Moser–Sutherland type [22–29].

One of the new features of our two-parameter deformation of the Heisenberg and parabose algebras is that the ordinary relations $[N, a] = -a$ and $[a_+, N] = -a_+$ are not satisfied in general, but become additively modified by the q-commutator of q^N with a and a_+. Even though the coordinate operator, the deformed derivative $D_{p,q}$ and the corresponding number operator obey those unmodified relations, it turns out that the deformation of the standard Hamiltonian $H = \frac{1}{2}\left(x^2 - D_{p,q}^2\right)$ in the (p, q)-deformed coordinate-momentum representation with $D_{p,q}$ in the place of the derivative is a function of another number operator from a different algebra with the modified relations. Motivated by this, we study the structure of Fock-like space related to the deformed Heisenberg algebra with dilation and the modified oscillator relations. The presence of modified oscillator relations results in a not ordinarily deformed Fock space representation (see [17–19] for comparison). We also solve the eigenvalue equation for the Hamiltonian H.

This paper is arranged as follows. In section 2, we introduce the (p, q)-deformed derivative, study its main properties and introduce the deformed Heisenberg algebra with dilation and a deformed parabose algebra. The basic properties of the new standard Hamiltonian and its connection to the number-like operator are described in section 3. The deformed Fock-like space structure and the eigenvalues of the corresponding number-like operator $\tilde{N}$ are derived in section 4. Finally, in section 5, we consider the eigenvalue equation for the Hamiltonian H, and obtain its formal solution.

2. A new kind of (p, q)-deformed oscillator

Starting from the single-mode parabose oscillator, we can write the parabose algebraic structure as the following trilinear commutation relations [5]:

$$[\{b, b^\dagger\}, b] = -2b, \qquad [b, \{b^\dagger, b^\dagger\}] = 4b^\dagger \tag{2.1}$$

where $b^\dagger$ and b are creation and annihilation operators of the paraboson, respectively, N is the number operator $N = \frac{1}{2}\{b, b^\dagger\} - \frac{p}{2}$ with the property $[N, b] = -b$, and $p = 1, 2, 3, \ldots$ is the order of paraquantization. Its standard representation in the Fock space involves the ket-vectors $|n\rangle \sim (b^\dagger)^n|0\rangle$, where $|0\rangle$ is the ground-state vector, which requires, for complete specification, not only the usual condition $a|0\rangle = 0$, but also the additional one

$$bb^\dagger|0\rangle = p|0\rangle. \tag{2.2}$$

From the above algebraic relations we have

$$\begin{aligned} bb^\dagger|2n\rangle &= [2n+1]_{p,-1}|2n\rangle, & bb^\dagger|2n+1\rangle &= [2n+2]_{p,-1}|2n+1\rangle, \\ b^\dagger b|2n\rangle &= [2n]_{p,-1}|2n\rangle, & b^\dagger b|2n+1\rangle &= [2n+1]_{p,-1}|2n+1\rangle, \end{aligned} \tag{2.3}$$

where the notation $[x]_{p,-1}$ stands for

$$[x]_{p,-1} = x + \frac{p-1}{2}(1 - (-1)^x), \tag{2.4}$$

and x can be any number or operator. Noticing $N|n\rangle = n|n\rangle$, from (2.3) we get

$$bb^\dagger = \tfrac{1}{2}(1 + (-1)^N)[N+1]_{p,-1} + \tfrac{1}{2}(1 - (-1)^N)[N+1]_{p,-1} = [N+1]_{p,-1}. \tag{2.5}$$

Similarly, we find

$$b^\dagger b = [N]_{p,-1}. \tag{2.6}$$

Addition and subtraction give $\{b, b^\dagger\} = 2N + p$ reproducing the definition of N, and

$$bb^\dagger - b^\dagger b = 1 + (p-1)(-1)^N, \tag{2.7}$$

which is a bilinear commutation relation of a paraboson [5] and can be regarded as a deformation of the ordinary Heisenberg algebra with p as a deformation parameter, and where $(-1)^N \equiv R$ is usually called the reflection operator with the properties $\{R, b\} = Rb + bR = 0$ and $R^2 = 1$.

Noticing that $[x]_{p,-1} = x + (p-1)[x]_{-1}$, where $[x]_{-1}$ is the Jackson q-number defined as $[x]_q = \frac{q^x-1}{q-1}$ for $q \neq 1$ and as x for $q = 1$ (see, for example, [21]), we introduce a more general (p, q)-number

$$[x]_{p,q} = x + \frac{p-1}{q-1}(q^x - 1) = x + (p-1)[x]_q, \tag{2.8}$$

which obviously reduces to (2.4) for $q = -1$. Corresponding to the new notation $[x]_{p,q}$, we also introduce a (p, q)-deformed derivative

$$D_{p,q} = D + (p-1)D_q, \tag{2.9}$$

where D is the ordinary derivative $\mathrm{d}/\mathrm{d}x$ and D_q is the Jackson q-derivative defined as

$$D_q f(x) = \frac{\mathrm{d}}{\mathrm{d}_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{2.10}$$

for $q \neq 1$ and as D for $q = 1$. When $p = 1$ we recover the usual derivative D, and when $q = 1$ we get $D_{p,1} = pD$. For the paraboson case $q = -1$ we get a differential–difference operator with reflection, which after a minor adjustment of the deformation parameters, is also known as a one-dimensional Dunkl operator [22–24].

So in this sense the operator (2.9) can be viewed as a two-parameter deformation of the one-dimensional Dunkl operator. We are grateful to Marcel de Jeu for bringing to our attention this interesting connection, certainly worth exploiting further, for example, in connection with deformations of multi-dimensional Dunkl operators.

Obviously, with the new derivative $D_{p,q}$ one has $D_{p,q}x^n = [n]_{p,q}x^{n-1}$ and $D_{p,q}\, e^{\alpha x}_{p.q} = \alpha e^x_{p,q}$, where the (p,q)-exponential is defined as $e^x_{p,q} = \sum_{n=0}^{\infty} x^n/[n]_{p,q}!$ with α being a constant and the (p,q)-factorial given by $[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q}$ with $[0]_{p,q}! \equiv 1$.

The coordinate operator $M_x : f(x) \mapsto xf(x)$, together with the derivative D and the Jackson q-derivative D_q, obey the Heisenberg and the q-deformed Heisenberg commutation relations [16], respectively:

$$DM_x - M_x D = I, \qquad D_q M_x - qM_x D_q = I. \tag{2.11}$$

From these relations it follows that the operators $D_{p,q}$, M_x and the multiplicative shift (rescaling) operator $R_q = q^{M_x D} = q^{xD} : f(x) \mapsto f(qx)$ and $M_x D = xD$ obey the following relations:

$$D_{p,q}M_x = [M_x D + 1]_{p,q}, \qquad M_x D_{p,q} = [M_x D]_{p,q}, \tag{2.12}$$

which lead to an important commutation relation

$$D_{p,q}M_x - M_x D_{p,q} = 1 + (p-1)R_q. \tag{2.13}$$

Obviously, for the $p = 1$ case it returns to the standard Heisenberg commutation relation and for the $q = -1$ case to the parabose commutation relation [7, 8]. Straightforward calculations give

$$R_q M_x R_q^{-1} = qM_x, \qquad R_q D R_q^{-1} = q^{-1}D, \qquad R_q D_{p,q} R_q^{-1} = q^{-1}D_{p,q}, \tag{2.14}$$

$$D_{p,q}M_x - qM_x D_{p,q} = (1-q)M_x D + p. \tag{2.15}$$

From the last relation we see that

$$M_x D = \frac{1}{1-q}([D_{p,q}, M_x]_q - p), \tag{2.16}$$

where $[u, v]_q = uv - qvu$ is the q-commutator. Note that (2.16) with (2.13) and (2.14) yields

$$[M_x D, D_{p,q}] = -D_{p,q}, \qquad [M_x D, M_x] = M_x, \tag{2.17}$$

as expected.

Motivated by these observations, we propose

$$aa_+ = [N+1]_{p,q}, \qquad a_+a = [N]_{p,q} \tag{2.18}$$

as the generalization of (2.5) and (2.6), leading to the relations

$$aa_+ - a_+a = 1 + (p-1)q^N, \tag{2.19}$$

of a new two-parameter deformation of the oscillator algebra. It can also be regarded as a q-deformed parabose algebra since for $q = -1$ it returns to (2.7). However, it is different from other q-deformed parabose algebras reported in the literature [6]. Here, the number operator N is defined via (2.19) and

$$aa_+ + a_+a = 2N + 1 + \frac{p-1}{q-1}(q^{N+1} + q^N - 2), \tag{2.20}$$

by the formula

$$N = \frac{1}{1-q}([a, a_+]_q - pI). \tag{2.21}$$

Note that (2.18), or more precisely its consequences (2.19) and (2.21), yields

$$[N, a] = -a + \frac{p-1}{q-1}(aq^N - q^{N+1}a) = -a + \frac{p-1}{q-1}[a, q^N]_q, \tag{2.22}$$

$$[a_+, N] = -a_+ + \frac{p-1}{q-1}(q^N a_+ - a_+ q^{N+1}) = -a_+ + \frac{p-1}{q-1}[q^N, a_+]_q. \quad (2.23)$$

This means that the properties of the classical oscillator

$$[N, a] = -a, \qquad [a_+, N] = -a_+ \quad (2.24)$$

hold here only under further special conditions, namely either when $p = 1$ or when $[a, q^N]_q = 0$ and $[q^N, a_+]_q = 0$, which can be written equivalently in 'quantum plane' intertwining form

$$aq^N = q^{N+1}a = qq^N a, \qquad q^N a_+ = a_+ q^{N+1} = qa_+ q^N, \quad (2.25)$$

or, using $q^{-N}q^N = I$ and $q^{N+1} = qq^N$, in the action form

$$q^N a q^{-N} = q^{-1}a, \qquad q^N a_+ q^{-N} = qa_+. \quad (2.26)$$

Moreover, relations (2.24) imply (2.25) and (2.26). So, when the fundamental relations (2.18) of the deformed oscillator hold, relations (2.24) are equivalent to (2.25).

From (2.21) we have $[a, a_+]_q = (1-q)N + pI$, and hence

$$[[a, a_+]_q, a] = (1-q)[N, a] = -(1-q)a - (p-1)[a, q^N]_q. \quad (2.27)$$

When (2.24) or (2.25) hold, relation (2.27) becomes

$$[[a, a_+]_q, a] = (1-q)[N, a] = -(1-q)a. \quad (2.28)$$

The parabose relation (2.1) is recovered for $q = -1$. So, we have obtained (p, q)-deformed (2.27) and q-deformed (2.28) parabose algebra.

3. Hamiltonian for q-deformed parabosonic system

In this section, we deduce some fundamental properties of the standard Hamiltonian element

$$H = \tfrac{1}{2}((a_+)^2 - a^2)$$

in the restricted q-deformed parabose algebra (2.18)–(2.21) with relations (2.24)–(2.26) and (2.28). The canonical example of operators satisfying these restricted relations, according to (2.11)–(2.17), is given by the operators $D_{p,q}$, M_x, the multiplicative rescaling operator R_q and $M_x D = xD$ in the sense of the representation $(a, a_+, N) \mapsto (D_{p,q}, M_x, M_x D)$. These properties play an important role for investigation of the corresponding eigenvalue problem which we consider in section 5.

We define $\tilde{a}$ and $\tilde{a}_+$ via the standard relations

$$\tilde{a} = \frac{a_+ + a}{\sqrt{2}}, \qquad \tilde{a}_+ = \frac{a_+ - a}{\sqrt{2}}. \quad (3.1)$$

After this transformation the Hamiltonian assumes another, also familiar, anticommutator form

$$H = \tfrac{1}{2}((a_+)^2 - a^2) = \tfrac{1}{2}(\tilde{a}\tilde{a}_+ + \tilde{a}_+\tilde{a}).$$

We require that the new number operator $\tilde{N}$ together with $\tilde{a}$ and $\tilde{a}_+$ satisfy the same relations as (2.18), namely

$$\tilde{a}\tilde{a}_+ = [\tilde{N} + 1]_{p,q}, \qquad \tilde{a}_+\tilde{a} = [\tilde{N}]_{p,q}. \quad (3.2)$$

Then, for $\tilde{a}$ and $\tilde{a}_+$, relations like (2.19), (2.20) hold

$$\tilde{a}\tilde{a}_+ - \tilde{a}_+\tilde{a} = 1 + (p-1)q^{\tilde{N}}, \tag{3.3}$$

$$\tilde{a}\tilde{a}_+ + \tilde{a}_+\tilde{a} = 2\tilde{N} + 1 + \frac{p-1}{q-1}(q^{\tilde{N}+1} + q^{\tilde{N}} - 2), \tag{3.4}$$

and hence

$$\tilde{N} = \frac{1}{1-q}([\tilde{a}, \tilde{a}_+]_q - pI) \tag{3.5}$$

$$= \frac{1}{2}\left(a_+^2 - a^2\right) + \frac{1+q}{2(1-q)}(aa_+ - a_+a) - \frac{p}{1-q}I$$

$$= H + \frac{1+q}{2(1-q)}(I + (p-1)q^N) - \frac{p}{1-q}I. \tag{3.6}$$

Since $\tilde{a}\tilde{a}_+ - \tilde{a}_+\tilde{a} = aa_+ - a_+a$, we get by (2.19) and (3.3) the equality

$$(p-1)q^{\tilde{N}} = (p-1)q^N \tag{3.7}$$

which is equivalent to $q^{\tilde{N}} = q^N$ when $p \neq 1$. So, by (3.6) and (3.7), we have

$$\tilde{N} = H + \frac{1+q}{2(1-q)}(I + (p-1)q^{\tilde{N}}) - \frac{p}{1-q}I, \tag{3.8}$$

$$H = \tilde{N} - \frac{1+q}{2(1-q)}(I + (p-1)q^{\tilde{N}}) + \frac{p}{1-q}I. \tag{3.9}$$

The equality (3.9) is of special interest for further investigation of H since it means that H is the function $F(x) = x - \frac{1+q}{2(1-q)}(I + (p-1)q^x) + \frac{p}{1-q}$ of $\tilde{N}$.

Note that

$$q^{\tilde{N}}\tilde{a}q^{-\tilde{N}} = \tfrac{1}{2}(q+q^{-1})\tilde{a} + \tfrac{1}{2}(q-q^{-1})\tilde{a}_+ = \cosh r\tilde{a} + \sinh r\tilde{a}_+ \qquad (q = e^r)$$

$$q^{\tilde{N}}\tilde{a}_+q^{-\tilde{N}} = \tfrac{1}{2}(q-q^{-1})\tilde{a} + \tfrac{1}{2}(q+q^{-1})\tilde{a}_+ = \sinh r\tilde{a} + \cosh r\tilde{a}_+,$$

which indicates that the usual relations $[\tilde{N}, \tilde{a}] = -\tilde{a}$, $[\tilde{a}_+, \tilde{N}] = -\tilde{a}_+$ are not in general satisfied in the new (p, q)-deformed oscillator, recalling our previous discussion on relations (2.24)–(2.27). So, the transformation of generators (3.1) is not allowed if we insist on preserving relation (2.28). Instead, the more general relations like (2.22), (2.23) and (2.27) hold:

$$[\tilde{N}, \tilde{a}] = -\tilde{a} + \frac{p-1}{q-1}(\tilde{a}q^{\tilde{N}} - q^{\tilde{N}+1}\tilde{a}) = -\tilde{a} + \frac{p-1}{q-1}[\tilde{a}, q^{\tilde{N}}]_q, \tag{3.10}$$

$$[\tilde{a}_+, \tilde{N}] = -\tilde{a}_+ + \frac{p-1}{q-1}(q^{\tilde{N}}\tilde{a}_+ - \tilde{a}_+q^{\tilde{N}+1}) = -\tilde{a}_+ + \frac{p-1}{q-1}[q^{\tilde{N}}, \tilde{a}_+]_q, \tag{3.11}$$

$$[[\tilde{a}, \tilde{a}_+]_q, \tilde{a}] = (1-q)[\tilde{N}, \tilde{a}] = -(1-q)\tilde{a} - (p-1)[\tilde{a}, q^{\tilde{N}}]_q. \tag{3.12}$$

It would be of interest to have a description of some general classes of transformations of generators preserving the q-deformed parabose relation (2.28). At this time, we do not know a satisfactory solution to this problem.

4. Fock space representation of the new oscillator

We have seen that the Hamiltonian $H = \frac{1}{2}((a_+)^2 - a^2)$ can be expressed as a function of the new number operator $\tilde{N}$ for $\tilde{a}$ and $\tilde{a}_+$ satisfying more general deformed oscillator relations (3.10). Motivated by this observation we study in this section the structure of the Fock space of the oscillator $(\tilde{a}, \tilde{a}_+, \tilde{N})$.

As usual, we will assume that firstly the new oscillator is acting on the separable Hilbert space with the inner product $\langle u|v\rangle$ and an orthonormal basis $\{|n\rangle\}$, $n = 0, 1, 2, 3, \ldots$, meaning that $\langle m|n\rangle = \delta_{m,n}$ is 1 if $m = n$ and 0 otherwise, and the completion relation $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$ holds; secondly at least the linear space spanned by $\{|n\rangle, n = 0, 1, 2, \ldots\}$ belongs to the domains of definition of $\tilde{a}$ and $\tilde{a}_+$ and thirdly the involution condition $\tilde{a}_+ = \tilde{a}^\dagger$ is satisfied in the sense that $\langle n|\tilde{a}_+|m\rangle = \langle n|\tilde{a}^\dagger|m\rangle = \overline{\langle m|\tilde{a}|n\rangle}$ for any two basis elements. Note that with the involution condition $\tilde{a}_+ = \tilde{a}^\dagger$ the operator $\tilde{N}$ is a normal operator meaning that $\tilde{N}\tilde{N}^\dagger = \tilde{N}^\dagger\tilde{N}$. Finally and most importantly for some scalars ε_n we assume

$$\tilde{N}|n\rangle = \varepsilon_n|n\rangle, \qquad \tilde{a}|0\rangle = 0, \qquad |n\rangle = \mathrm{e}^{\mathrm{i}\theta_n} c_n \tilde{a}^{\dagger n}|0\rangle$$

where c_n is positive and θ_n is a real number for any $n = 0, 1, 2, 3, \ldots$. Note that the involution condition $\tilde{a}_+ = \tilde{a}^\dagger$ and (3.2) imply that $[\varepsilon_n]_{p,q}$ and $[\varepsilon_n + 1]_{p,q}$ should be nonnegative numbers. If we do not assume any involution condition, then there is no such extra restriction on ε_n.

In order to find the eigenvalues ε_n of $\tilde{N}$, note that using (3.2) we get

$$(1-q)\tilde{N}\tilde{a}^\dagger = (\tilde{a}\tilde{a}^\dagger - q\tilde{a}^\dagger\tilde{a} - pI)\tilde{a}^\dagger = ([N+1]_{p,q}\tilde{a}^\dagger - q\tilde{a}^\dagger[N+1]_{p,q} - p\tilde{a}^\dagger),$$

$$(1-q)\varepsilon_{n+1}|n+1\rangle = (1-q)\tilde{N}|n+1\rangle = ([\varepsilon_{n+1}+1]_{p,q} - q[\varepsilon_n+1]_{p,q} - p)|n+1\rangle,$$

resulting in the implicit recurrence equations for $\{\varepsilon_n\}$, $n = 0, 1, 2, 3, \ldots$:

$$[\varepsilon_{n+1}+1]_{p,q} - (1-q)\varepsilon_{n+1} = q[\varepsilon_n+1]_{p,q} + p, \tag{4.1}$$

which is equivalent to $[\varepsilon_{n+1}]_{p,q} = [\varepsilon_n + 1]_{p,q}$ for $q \neq 0$. For ε_0, we have

$$\begin{aligned} (1-q)\varepsilon_0|0\rangle &= (1-q)\tilde{N}|0\rangle = (\tilde{a}\tilde{a}^\dagger - q\tilde{a}^\dagger\tilde{a} - pI)|0\rangle \\ &= ([\tilde{N}+1]_{p,g} - pI)|0\rangle = ([\varepsilon_0+1]_{p,g} - p)|0\rangle, \\ (1-q)\varepsilon_0 &= [\varepsilon_0+1]_{p,g} - p, \end{aligned} \tag{4.2}$$

or equivalently

$$[\varepsilon_0]_{p,g} = \varepsilon_0 + (p-1)[\varepsilon_0]_q = 0. \tag{4.3}$$

When $q = 1$ we get $p\varepsilon_0 = 0$ thus for $p \neq 0$ recovering the classical result $\varepsilon_0 = 0$ for the ground state of the usual quantum harmonic oscillator. Note that the equation can also be written as $(p-1)q^{\varepsilon_0} = -(q-1)\varepsilon_0 + (p-1)$. But, in this form we do not get any information on ε_0 for the case $q = 1$.

The implicit recurrence relations (4.1) can also be written as

$$[\varepsilon_{n+1}+1]_{p,q} - (1-q)\varepsilon_{n+1} - q([\varepsilon_n+1]_{p,q} - (1-q)\varepsilon_n) = q(1-q)\varepsilon_n + p. \tag{4.4}$$

This equation expresses a nonlinear function of ε_{n+1} via a nonlinear function of ε_n. But, after taking the convolution of both sides with (q^k), $k = 0, \ldots, n$, and making a telescope summation on the left-hand side of this equation we get

$$[\varepsilon_{n+1}+1]_{p,q} - (1-q)\varepsilon_{n+1} = (1-q)q\sum_{k=0}^{n} \varepsilon_{n-k}q^k + [n+2]_q p, \tag{4.5}$$

expressing a nonlinear function of ε_{n+1} as a linear combination of $1, \varepsilon_0, \ldots, \varepsilon_n$. In particular, for $n = 0$ we get $[\varepsilon_1 + 1]_{p,q} - (1-q)\varepsilon_1 = (1-q)q\varepsilon_0 + [2]_q p$. For $q = 1$ or $p = 1$, we get $\varepsilon_1 = 1$ as expected.

In a sense the sequence $(\varepsilon_k), k = 0, 1, \ldots,$ can be viewed as a new number system generalizing or deforming the natural numbers. The defining recurrence equations for these numbers are not algebraic except for some special values of the deformation parameters. Of course while these recurrence equations can be used to calculate the approximations of eigenvalues (ε_k), $k = 0, 1, \ldots$ numerically, it could also be important to get deeper insight into number theoretical, structural and combinatorial properties of these new numbers so intimately connected with deformed oscillators and parastatistics.

In order to specify completely the action of the operators $\tilde{a}$ and $\tilde{a}^\dagger$, it is left to find the normalizing constants $(c_n, n = 0, 1, 2, \ldots)$ because

$$\tilde{a}|n+1\rangle = \exp(\mathrm{i}(\theta_{n+1} - \theta_n))\frac{c_{n+1}}{c_n}\tilde{a}\tilde{a}^\dagger|n\rangle = \exp(\mathrm{i}(\theta_{n+1} - \theta_n))\frac{c_{n+1}}{c_n}[\varepsilon_n + 1]_{p,q}|n\rangle \tag{4.6}$$

$$\tilde{a}^\dagger|n\rangle = \exp(\mathrm{i}(\theta_n - \theta_{n+1}))\frac{c_n}{c_{n+1}}|n+1\rangle, \qquad \text{for} \quad n = 0, 1, 2, \ldots. \tag{4.7}$$

Using the normalization condition $\langle n+1|n+1\rangle = 1$ and the involution condition we get

$$c_{n+1}^2 = \left(\prod_{k=0}^{n}[\varepsilon_k + 1]_{p,q}\right)^{-1}, \qquad \left(\frac{c_{n+1}}{c_n}\right)^2 = [\varepsilon_n + 1]_{p,q}^{-1}, \tag{4.8}$$

$$\tilde{a}|n+1\rangle = \exp(\mathrm{i}(\theta_{n+1} - \theta_n))[\varepsilon_n + 1]_{p,q}^{\frac{1}{2}}|n\rangle, \tag{4.9}$$

$$\tilde{a}^\dagger|n\rangle = \exp(\mathrm{i}(\theta_n - \theta_{n+1}))[\varepsilon_n + 1]_{p,q}^{\frac{1}{2}}|n+1\rangle. \tag{4.10}$$

5. An eigenvalue equation for q-deformed parabosonic system

In the Fock representations (4.3), (4.4), the standard Hamiltonian $H = \frac{1}{2}((a_+)^2 - a^2)$ is diagonal on the basis $\{|n\rangle\}$, namely $H|n\rangle = E_n|n\rangle$ with eigenvalue $E_n = \frac{1+q}{2}[\varepsilon_n]_{p,q} + \frac{1-q}{2}\varepsilon_n + \frac{p}{2}$ as follows from (3.9). In this section, we consider an eigenvalue equation for the Hamiltonian H in the coordinate-momentum representation of the q-deformed parabose algebra (2.28).

According to (2.11)–(2.17), the operators $D_{p,q}$, M_x, the multiplicative rescaling operator R_q and $M_x D = xD$ obey all the commutation relations (2.18)–(2.28) in the sense of the representation $(a, a_+, N) \mapsto (D_{p,q}, M_x, M_x D)$. So, the operators $D_{p,q}$ and M_x can be viewed as an analogue, for the new oscillator, of the canonical coordinate-momentum representation of the non-deformed Heisenberg commutation relation. In this representation, the Hamiltonian becomes

$$H_c = \tfrac{1}{2}((a_+)^2 - a^2) = \tfrac{1}{2}x^2 - \tfrac{1}{2}D_{p,q}^2, \tag{5.1}$$

and we consider the eigenvalue problem

$$H_c\psi(x) = \tfrac{1}{2}x^2\psi(x) - \tfrac{1}{2}D_{p,q}^2\psi(x) = \lambda\psi(x). \tag{5.2}$$

Note that in the canonical representation $(a, a_+, N) \mapsto (D_{p,q}, M_x, M_x D)$ we have

$$\tilde{a} = \frac{M_x + D_{p,q}}{\sqrt{2}}, \qquad \tilde{a}_+ = \frac{M_x - D_{p,q}}{\sqrt{2}}, \tag{5.3}$$

$$\tilde{N} = H + \frac{1+q}{2(1-q)}(I + (p-1)R_q) - \frac{p}{1-q}I = H - \frac{p}{2}I - \frac{(1+q)(p-1)}{2}M_x D_q. \tag{5.4}$$

The factorization of solution to a product of a polynomial and a Gaussian-like term, so nicely resolving the problem for the case of the non-deformed oscillator and for the parabose case $q = -1$, seems to be difficult if not impossible for the general q. We will look here for a solution, without making any prior factorizations, directly in the power series form $\psi(x) = \sum_{k=0}^{\infty} h_k x^k$. Substituting this series into (5.2) we get the following recurrence equations for the coefficients:

$$h_{k-2} - h_{k+2}[k+2]_{p,q}[k+1]_{p,q} = 2\lambda h_k, \qquad k = 2, 3, 4, \ldots, \tag{5.5}$$

$$h_2[2]_{p,q}[1]_{p,q} = -2\lambda h_0, \qquad h_3[3]_{p,q}[2]_{p,q} = -2\lambda h_1. \tag{5.6}$$

Multiplication of (5.5) by appropriate coefficients and telescopic summation on the left-hand side gives

$$\begin{aligned} h_{k-2-4l} &- h_{k+2}\prod_{j=0}^{l}[k+2-4j]_{p,q}[k+1-4j]_{p,q} \\ &= 2\lambda \sum_{s=0}^{l} h_{k-4s} \prod_{j=0}^{l-s-1} [k+2+4(j-l)]_{p,q}[k+1+4(j-l)]_{p,q}. \end{aligned} \tag{5.7}$$

Putting, for example, $k = 4l + 2$ we get

$$\begin{aligned} h_0 &- h_{4(l+1)}\prod_{j=0}^{l}[4(l-j+1)]_{p,q}[4(l-j+1)-1]_{p,q} \\ &= 2\lambda \sum_{s=0}^{l} h_{4(l-s)+2} \prod_{j=0}^{l-s-1} [4(l+j)]_{p,q}[4(l+j)-1]_{p,q}. \end{aligned} \tag{5.8}$$

Solutions of the recurrence equations (5.5), (5.6) can be easily expressed in terms of continued fractions as follows:

$$h_{2k} = h_0\left(\prod_{l=1}^{k} \gamma_{2l}^{-1}\right), \qquad h_{2k+1} = h_1\left(\prod_{l=1}^{k} \gamma_{2l+1}^{-1}\right),$$

$$\begin{aligned} \gamma_{2k} &= \alpha_{2k}(-2\lambda + \alpha_{2(k-1)}(\cdots + \alpha_{2(k-j)}(\cdots + \alpha_2(-2\lambda+\gamma_0)^{-1})^{-1}\cdots)^{-1} \\ &= g_{2k}\circ g_{2(k-1)}\circ\cdots\circ g_{2(k-j)}\circ\cdots\circ g_2(0), \qquad \gamma_0 = 0 \\ \gamma_{2k+1} &= \alpha_{2k+1}(-2\lambda + \alpha_{2(k-1)+1}(\cdots + \alpha_{2(k-j)+1}(\cdots + \alpha_3(-2\lambda+\gamma_1)^{-1})^{-1}\cdots)^{-1} \\ &= g_{2k+1}\circ g_{2(k-1)+1}\circ\cdots\circ g_{2(k-j)+1}\circ\cdots\circ g_3(0), \qquad \gamma_1 = 0 \end{aligned}$$

$$g_{2j}(x) = \alpha_{2j}(-2\lambda + x)^{-1}, \qquad g_{2j+1}(x) = \alpha_{2j+1}(-2\lambda + x)^{-1}, \qquad j = 1, 2, 3, \ldots,$$

$$\alpha_{k+2} = [k+2]_{p,q}[k+1]_{p,q}, \qquad k = 0, 1, 2, 3, \ldots.$$

It should of course be of great interest to find further expressions of h_k in terms of some known deformed special and combinatorial functions, study convergence and other analytic properties of ψ and also describe solutions for non-generic values of parameters p and q leading to division by zero in the formulae for h_k. We hope to address these problems in further publications.

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